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The hybrid spectral problem and Robin boundary conditions

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Abstract

The hybrid spectral problem where the field satisfies Dirichlet conditions (D) on part of the boundary of the relevant domain and Neumann (N) on the remainder is discussed in simple terms. A conjecture for the C_1 coefficient is presented and the conformal determinant on a 2-disc, where the D and N regions are semi-circles, is derived. Comments on higher coefficients are made. A separable second-order hemisphere hybrid problem is introduced that involves Robin boundary conditions and leads to logarithmic terms in the heat-kernel expansion which are evaluated explicitly.

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1. Introduction

The explicit construction of the general form of the heat-kernel expansion coefficients has reached the stage when further progress is impeded mainly by ungainliness. Unless there is some compelling reason for finding a specific higher coefficient, its exhibition is not particularly enlightening and is not really worth the, often considerable, effort. Other, more productive, avenues consist of generalizing the differential operator, the manifold or the boundary conditions. In the latter context a simply stated extension is the class of problems where the field satisfies Dirichlet conditions (D), say, on part of the boundary and Neumann (N) on the remainder. These boundary conditions are sometimes termed 'mixed' in the classical literature (e.g. Sneddon [1]) or sometimes 'hybrid' (e.g. Treves [2] chapter 37). A brief history of the corresponding potential theory, sometimes referred to as the Zaremba problem, is contained in Azzam and Kreysig [3]. It is also interesting to note that these conditions have occurred in string theory, [4], and have recently been considered in connection with isospectrality, Jakobson *et al* [5].

To set some notation, the conventional short-time expansion of the integrated heat kernel of a smooth boundary value problem is

$$K(t) \equiv \mathrm{Tr}\, \mathrm{e}^{-Pt} \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0,1/2,1\ldots}^{\infty} C_n t^n, \tag{1}$$

where P is a smooth (singularity free) elliptic differential operator and where, initially, the manifold, its boundary and any boundary conditions are all smooth. Typically P is the Laplacian, plus possibly a smooth potential, and the coefficients are locally computed as integrals over the manifold, or its boundary, of local geometrical invariants constructed from the curvature, for example. See [6] for an extensive treatment.

Any relaxation of smoothness can result in modifications to this expansion. For example, a singular potential can lead to 'anomalous' powers of t, e.g. [7].

The hybrid N/D case has non-smooth boundary conditions and so can be classed as a singular boundary value problem, see, e.g., [8, 9]. Even though it may be geometrically smooth, the codimension-2 region, Σ, where the D and N conditions meet, can be regarded as a conical singularity. The general existence of logarithmic terms in the expansion of $K(t)$ for such singularities, and other situations, has been analysed for a long time (e.g. Cheeger [10], Brüning and Seeley [11], Grubb and Seeley [12], Grubb [13, 14], Gilkey and Grubb [15], Seeley [16]).

Seeley [17], based on [11], has proved the existence of an asymptotic expansion that allows for logarithmic terms but then shows for a particular N/D situation that such terms do not appear. A classic case is the simple π-wedge where the explicit calculation shows that logarithmic terms are absent.

It is anticipated that the heat-kernel coefficients will receive contributions from Σ. This has been confirmed by Avramidi [8, 9] and work by van den Berg and Gilkey [18], on heat *content* is also pertinent.

In this paper I wish to discuss some aspects of the hybrid question that are mainly example driven and with a minimum of algebra. It is hoped that these considerations will prove useful in more general field and string theoretic areas where heat-kernel coefficients play important roles in divergence and scaling questions. My approach is mostly global, as opposed to the local treatment by Avramidi [8, 9].

I begin with the Laplacian eigenproblem on the interval with D and N conditions on the ends. This is then embedded in higher dimensions and used to determine the C_1 hybrid coefficient for the 2-wedge from the purely D, or purely N, expressions, which are very old. This result is then used to write down the general C_1 for a d-manifold with piecewise smooth boundary and a conjecture is made for the case where N is replaced by Robin conditions, denoted (R). A crude check in the N–D case results from applying the technique to the 2-lune.

Although my main attention is directed at C_1, some very limited information on the higher coefficients, $C_{3/2}$ and C_2, is extracted in section 5 from the hybrid half-disc.

As an example of the use of the conjectured form of the hybrid C_1, I evaluate the Laplacian functional determinant on the N–D disc in section 6.

In section 7, I set up a separable Robin hybrid problem for the Laplacian on the 2-sphere and show in later sections that logarithmic terms appear in the expansion of the heat kernel. Perturbation theory is used to bolster confidence in the existence of the model.

2. Basic idea

For the Laplacian, simple calculation, or the drawing of a few modes, shows that on the interval of length L with Dirichlet (D) and Neumann (N) conditions, the spectral data of the various eigenproblems are related by

$$(D, N)_L \cup (D, D)_L = (D, D)_{2L} \qquad (D, N)_L \cup (N, N)_L = (N, N)_{2L} \tag{2}$$

$$(D, D)_L \cup (N, N)_L = P_{2L}, \tag{3}$$

where the notation (D, N) signifies a problem with D conditions at one end and N at the other and P stands for periodic conditions. Averaging (2) gives, using (3)

$$(D, N)_L \cup \tfrac{1}{2} P_{2L} = \tfrac{1}{2} P_{4L}. \tag{4}$$

These relations can be extended algebraically to any spectral quantity, such as the heat kernel.

The 'subtraction' implied by (2) and (4), in order to extract the (D, N) part, amounts to a cull of the even modes on the doubled interval, as is well known (cf Rayleigh [19], vol I, p 247).

The relations can be applied to the arc of a circle, which might form part of an $SO(2)$ foliation of a two-dimensional region (or the projection of a higher dimensional region onto two dimensions). A wedge is a good example which I will now look at. Separability of the modes implies that the relations (2) apply equally well to the wedge, where the notation signifies that either D or N holds on the straight sides (say $\theta = 0$ and $\theta = \beta$). Equation (2) can then immediately be applied to the heat-kernel, and its small-time expansion, to determine the form of the heat-kernel coefficients in the (D, N) combination. I will show how this works out for the C_1 coefficient.

The (D, D) and (N, N) wedge coefficients are well known and have been derived in several ways. They are

$$C_1^{\text{wedge}}(D, D) = C_1^{\text{wedge}}(N, N) = \frac{\pi^2 - \beta^2}{6\beta}. \tag{5}$$

Hence from (2)

$$C_1^{\text{wedge}}(D, N) = -\frac{\pi^2 + 2\beta^2}{12\beta}. \tag{6}$$

This last result has been derived by Watson in a rather complicated way using the modes directly [20].

Incidentally the conjecture by Gottlieb (equation (3.5) in [21]), that the (N, D) case differs from the (D, D) one only by a sign, is incorrect, although it is true in the special case of a right-angled wedge, as is easily checked by looking at rectilinear flat domains. The statement is carried through into [22].

Sommerfeld, [23] vol 2, p 827, also mentions the 'mixed' wedge and indicates how to treat it using images if $\beta = \pi n/m$.

It is useful to note that, as pointed out by Cheeger [10] p 605, expressions (5) and (6) are not locally computable geometric invariants, as evidenced by the $1/\beta$ dependence. (Seeley [17], in his context, refers to such nonlocal terms as 'semi-global'. See also [9].) In the derivation of the wedge coefficients by Cheeger (see Bordag *et al* [24], Cognola and Zerbini [25]) the $1/\beta$ terms arise from the β-interval ζ-function evaluated at the argument $-1/2$. As noted in [24] this is the Casimir energy on the interval, clarifying the nonlocal character of this term. By contrast, the term proportional to β is locally computable.

It is an important technical point that, as discussed by Avramidi [8, 9], and Seeley [17], it is necessary to specify boundary conditions at the singular region Σ to give a well-defined problem. The derivations of expressions (5) assume Dirichlet at the wedge apex. This amounts to taking the Friedrichs extension by default and extends to the hybrid wedge (6) by (2). I will continue with this simplifying choice for the rest of this paper.

3. The general case

Consider in general dimension a manifold whose boundary is piecewise smooth consisting of domains, $\partial\mathcal{M}_i$, which intersect in codimension-2 manifolds, $\mathcal{I}_{ij}$. On each of the pieces, $\partial\mathcal{M}_i$, either D or N is imposed.

In general dimension, for all D or all N, the smeared coefficients are known,

$$C_1(D) = \left(\frac{1}{6} - \xi\right)\int_{\mathcal{M}} Rf\,\mathrm{d}V + \int_{\partial\mathcal{M}}\left(\frac{1}{3}\kappa - \frac{1}{2}n\partial\right) f\,\mathrm{d}A + \frac{1}{6}\int_{\mathcal{I}}\frac{\pi^2-\beta^2}{\beta} f\,\mathrm{d}L \tag{7}$$

$$C_1(N) = \left(\frac{1}{6} - \xi\right)\int_{\mathcal{M}} Rf\,\mathrm{d}V + \int_{\partial\mathcal{M}}\left(\frac{1}{3}\kappa + \frac{1}{2}n\partial\right) f\,\mathrm{d}A + \frac{1}{6}\int_{\mathcal{I}}\frac{\pi^2-\beta^2}{\beta} f\,\mathrm{d}L, \tag{8}$$

where $\partial\mathcal{M}$ is the union of the boundary pieces and $\mathcal{I}$ that of the intersections,

$$\partial\mathcal{M} = \bigcup_i \partial\mathcal{M}_i, \qquad \mathcal{I} = \bigcup_{i<j}\mathcal{I}_{ij}.$$

The 'smeared' coefficients result from the trace $\mathrm{Tr}\, f\,\mathrm{e}^{-Pt}$, where f is a spatially local operator, and are handy when discussing conformal transformations and for extracting the integrands. I will not use this freedom in any overt calculational way (see Kirsten [26]).

For a mixture of D and N, the volume contribution clearly remains unchanged while the surface contribution divides simply into a sum separately over those regions $\partial\mathcal{M}(D)$ and $\partial\mathcal{M}(N)$ subject to D and N, respectively. The codimension-2 intersections $\mathcal{I}_{ij}$ divide into the three (wedge) types $\mathcal{I}(D,D)$, $\mathcal{I}(N,N)$ and $\mathcal{I}(N,D)$ so the corresponding C_1 is, using (6),

$$\begin{aligned} C_1(D,N) = {} & \left(\frac{1}{6} - \xi\right)\int_{\mathcal{M}} Rf\,\mathrm{d}V + \int_{\partial\mathcal{M}(D)}\left(\frac{1}{3}\kappa - \frac{1}{2}n\partial\right) f\,\mathrm{d}A + \int_{\partial\mathcal{M}(N)}\left(\frac{1}{3}\kappa + \frac{1}{2}n\partial\right) f\,\mathrm{d}A \\ & + \frac{1}{6}\int_{\mathcal{I}(D,D)\cup\mathcal{I}(N,N)}\frac{\pi^2-\beta^2}{\beta} f\,\mathrm{d}L - \frac{1}{12}\int_{\mathcal{I}(D,N)}\frac{\pi^2+2\beta^2}{\beta} f\,\mathrm{d}L. \end{aligned} \tag{9}$$

In accordance with a previous remark, the wedge-like codimension-2 contributions in the above expressions are not locally computable.

If the boundary is smooth, then all the dihedral angles β equal π and the codimension-2 part of (9) (the last two integrals) reduces to

$$-\frac{\pi}{4}\int_{\mathcal{I}(D,N)} f\,\mathrm{d}L. \tag{10}$$

To repeat, even though the boundary is smooth, the region $\mathcal{I}(D,N) \equiv \Sigma$ is a singular region.

For example, for the 3-ball with D on the northern hemisphere and N ($S = 0$) on the southern,

$$C_1(D,N) = \frac{8\pi}{3} - \frac{\pi^2}{2},$$

for the smearing function, $f = 1$.

Expressions (7), (8) and (9) are in accord with Kac's principle of not feeling the boundary [27, 28], which implies that, C_1 will take contributions from the manifolds of codimension zero, one and two independently.

A local derivation of (10), justifying Kac's principle, has been given by Avramidi [8, 9]. It has also been obtained by van den Berg (unpublished).

For later reference, I would like to extend the Neumann conditions to Robin, (R), ones,

$$(n\partial - S)\Phi|_{\partial\mathcal{M}} = 0, \tag{11}$$

where n is the inward normal and S can depend on position. For example, in place of (8) one might expect,

$$C_1(R) = \left(\frac{1}{6} - \xi\right) \int_{\mathcal{M}} Rf \, \mathrm{d}V + \int_{\partial\mathcal{M}} \left(\frac{1}{3}\kappa - 2S + \frac{1}{2}n\partial\right) f \, \mathrm{d}A + \frac{1}{6} \int_{\mathcal{I}} \frac{\pi^2 - \beta^2}{\beta} f \, \mathrm{d}L. \quad (12)$$

The first two terms are the standard ones, e.g. [6, 29], for a smooth boundary. The last, codimension-2, term has actually not been derived directly for the Robin wedge but it holds when $S = 0$ and dimensions show that the Robin function cannot enter algebraically into this term.

On the same basis my conjecture for the corresponding $C_1(D, R)$ is

$$\begin{aligned} C_1(D, R) = {} & \left(\frac{1}{6} - \xi\right) \int_{\mathcal{M}} Rf \, \mathrm{d}V + \int_{\partial\mathcal{M}(D)} \left(\frac{1}{3}\kappa - \frac{1}{2}n\partial\right) f \, \mathrm{d}A \\ & + \int_{\partial\mathcal{M}(R)} \left(\frac{1}{3}\kappa - 2S + \frac{1}{2}n\partial\right) f \, \mathrm{d}A + \frac{1}{6} \int_{\mathcal{I}(D,D)\cup\mathcal{I}(R,R)} \frac{\pi^2 - \beta^2}{\beta} f \, \mathrm{d}L \\ & - \frac{1}{12} \int_{\mathcal{I}(D,R)} \frac{\pi^2 + 2\beta^2}{\beta} f \, \mathrm{d}L. \end{aligned} \quad (13)$$

Proceeding on the basis that (13) is correct, an expression for $C_1(N, R)$ can be obtained by making $\partial\mathcal{M}(D)$ empty and dividing $\partial\mathcal{M}(R)$ into $\partial\mathcal{M}(N) \cup \partial\mathcal{M}(R)$. The conjecture is then,

$$\begin{aligned} C_1(N, R) = {} & \left(\frac{1}{6} - \xi\right) \int_{\mathcal{M}} Rf \, \mathrm{d}V + \int_{\partial\mathcal{M}} \left(\frac{1}{3}\kappa + \frac{1}{2}n\partial\right) f \, \mathrm{d}A \\ & - 2 \int_{\partial\mathcal{M}(R)} Sf \, \mathrm{d}A + \frac{1}{6} \int_{\mathcal{I}} \frac{\pi^2 - \beta^2}{\beta} f \, \mathrm{d}L. \end{aligned} \quad (14)$$

It is clear that the above expressions have a specific validity, even without the codimension-2 parts. Thus, although the Robin form (12) trivially reduces to the Neumann one, (8), when $S = 0$, it is not possible to obtain the Dirichlet form, (7), directly by setting $S = \infty$. The exhibited forms, which, to repeat, are coefficients in a 'small time' asymptotic expansion, are really valid in the limit of small $S^2 t$, as discussed by Fulling [30]. I return more specifically to Robin conditions from section 7 onwards.

4. The lune

The expression for $C_1(D, N)$ can be checked in a curved space case by considering a lune segment of a sphere.

Relations (2), (3) can be applied to the 2-lune where the intervals are the sections of the lines of latitude cut out by the two longitudes, $\phi = 0, \phi = \beta$. In this case the extrinsic curvatures vanish (the boundaries are geodesically embedded) but there is a volume (area) term independent of the boundary conditions.

The ζ-functions are now somewhat more explicit [31, 32]. It is possible to work with a general angle, β, but I choose $\beta = \pi/q, q \in \mathbb{Z}$. The ζ-functions have been derived in [32] and used in [33].

Denoting the lune by $\mathcal{L}(\beta)$ one has

$$(D, N)_{\mathcal{L}(\beta)} \cup (D, D)_{\mathcal{L}(\beta)} = (D, D)_{\mathcal{L}(2\beta)}, \qquad (D, N)_{\mathcal{L}(\beta)} \cup (N, N)_{\mathcal{L}(\beta)} = (N, N)_{\mathcal{L}(2\beta)},$$

so that the corresponding ζ-functions combine algebraically,

$$\zeta_\beta^{ND}(s) = \zeta_{2\beta}^{DD}(s) - \zeta_\beta^{DD}(s) = \zeta_{2\beta}^{NN}(s) - \zeta_\beta^{NN}(s). \quad (15)$$

The DD and NN ζ-functions have been derived in [32] as Barnes ζ-functions for conformal coupling in three dimensions (leading to simple eigenvalues) and yield the specific value, for example,

$$\zeta_\beta^{DD}(0) = \frac{1}{12}\left(\frac{\pi}{\beta} - \frac{\beta}{2\pi}\right),$$

which can be used to confirm expression (6) using the relation between C_1 and $\zeta(0)$. (In this case there are no zero modes.)

The volume contribution, $\beta/6$, to C_1 is standard and is the same for all boundary conditions. Hence the contribution of each (N, D) corner (of which there are two) is

$$\frac{1}{2}\left[-\frac{4\pi}{24}\left(\frac{\pi}{\beta} + \frac{\beta}{\pi}\right) - \frac{\beta}{6}\right] = -\frac{\pi^2 + 2\beta^2}{12\beta}$$

as required for the check.

5. The disc and semi-circle. Higher coefficients

The fact that the extrinsic curvatures are zero means that the lune is not excessively helpful in deriving the form of the higher coefficients in the (D, N) case. Some further information can be obtained by looking at the half-disc with semi-circular boundary having different conditions on the diameter and circumference.

A straightforward application of, say, the Stewartson and Waechter Laplace transform technique combined with an image method soon yields the results for the short-time expansions

$$K_{DD}(t) \sim \frac{1}{8t} - \frac{2+\pi}{8\sqrt{\pi t}} + \frac{5}{24} + \frac{\sqrt{t}(\pi + 16)}{256\sqrt{\pi}} + \left(\frac{1}{315} + \frac{1}{32}\right)t + \cdots \tag{16}$$

$$K_{ND}(t) \sim \frac{1}{8t} + \frac{2-\pi}{8\sqrt{\pi t}} - \frac{1}{24} + \frac{\sqrt{t}(\pi - 16)}{256\sqrt{\pi}} + \left(\frac{1}{315} - \frac{1}{32}\right)t + \cdots \tag{17}$$

$$K_{NN}(t) \sim \frac{1}{8t} + \frac{2+\pi}{8\sqrt{\pi t}} + \frac{5}{24} + \frac{\sqrt{t}(5\pi + 48)}{256\sqrt{\pi}} + \left(\frac{1}{45} + \frac{3}{32}\right)t + \cdots \tag{18}$$

$$K_{DN}(t) \sim \frac{1}{8t} - \frac{2-\pi}{8\sqrt{\pi t}} - \frac{1}{24} + \frac{\sqrt{t}(5\pi - 48)}{256\sqrt{\pi}} + \left(\frac{1}{45} - \frac{3}{32}\right)t + \cdots \tag{19}$$

where DN means D on the diameter and N on the circumference, etc.

The constant terms check with (5) and (6) for $\beta = \pi/2$. Also (3), applied to the diameter as a wedge of angle π, yields the D and N (e.g. [34]), *full* disc expansions.

The extrinsic curvature vanishes on the diameter and equals one on the circumference part of the boundary so some information on the $C_{3/2}$ and C_2 coefficients can be inferred. Formulae in the non-mixed types (D, D) and (N, N) have been given in [33, 35] which agree with the relevant parts of the above expressions. Indeed I used the hemi-disc in deriving these results.

Also in [33] will be found an expression for C_2 in the case the boundary parts $\partial\mathcal{M}_i$ are subject to Robin conditions with different boundary functions, S_i although all dihedral angles are restricted to $\pi/2$.

In the case of C_2, the $1/315$ is the contribution of the curved D semicircle while the $\pm 1/32$ is the effect of the (two) corners and likewise regarding the $1/45 \pm 3/32$ combination. The $C_{3/2}$ coefficient exhibits a similar structure. Experience with the flat wedge shows that it is unwise to draw too many conclusions when the angle is $\pi/2$. What we can say, however, is that, using the 3/2 coefficient as an exemplar, one term will have the general form

$$-\frac{\sqrt{\pi}}{24}\left[\int_{\mathcal{I}(D,D)}\lambda_{DD}(\beta)(\kappa_1+\kappa_2)\,\mathrm{d}L+\int_{\mathcal{I}(N,N)}\lambda_{NN}(\beta)(\kappa_1+\kappa_2)\,\mathrm{d}L\right.$$
$$\left.+\int_{\mathcal{I}(N,D)}(\lambda_{ND}(\beta)\kappa_D+\lambda_{DN}(\beta)\kappa_N)\,\mathrm{d}L\right]$$

where $\lambda_{ND}(\pi/2) = -\lambda_{DD}(\pi/2) = 3$ and $\lambda_{DN}(\pi/2) = -\lambda_{NN}(\pi/2) = -9$. This change of sign is a simple consequence of images, or of (2) since the DD and NN quantities vanish when $\beta = \pi$.

6. The disc determinant

A direct attack via modes, of what is, after rectilinear domains, the simplest two-dimensional situation, i.e. a disc subject to N on one half of the circumference, and D on the rest, would seem to be difficult in so far as the construction of the ζ-function or heat kernel is concerned. However, the functional determinant, defined conventionally as $\exp(-\zeta'(0))$, appears to be accessible by conformal transformation from that on an ND-lune of angle π, i.e. a hemisphere with N on one half of the rim (the equator) and D on the rest, which is an easy quantity to find in terms of Barnes ζ-function from (15).

Instead of the determinant I use the effective action, W, defined by $W = -\zeta'(0)/2$. Integrating the conformal anomaly leads to the relation,

$$W[\mathrm{e}^{-2\omega}g] = W[g] + W[\mathrm{e}^{-2\omega}g, g],$$

where $W[\mathrm{e}^{-2\omega}g, g]$ is the cocyle function.

For the above programme to work, one would need the conjectured form of C_1, (13), to be valid in order to construct the required cocycle function in two dimensions. Applying the standard techniques this is (cf [36]), for a *smooth* boundary,

$$W[\mathrm{e}^{-2\omega}g,g] = \frac{1}{24\pi}\int_{\mathcal{M}}\omega(R+\Box\omega)\,\mathrm{d}V+\frac{1}{12\pi}\int_{\partial\mathcal{M}}\omega\left(\kappa+\frac{1}{2}(n\partial)\omega\right)\mathrm{d}A$$
$$+\frac{1}{8\pi}\left(\int_{\partial\mathcal{M}(N)}-\int_{\partial\mathcal{M}(D)}\right)(n\partial)\omega\,\mathrm{d}A-\frac{1}{16}\sum_k\omega_k, \tag{20}$$

where k labels the points where D and N meet and ω_k are the values of ω at these points. If $\partial\mathcal{M}(D)$ is empty there is a volume term coming from the pure N zero mode.

To go from the hemisphere to the disc I employ the equatorial stereographic projection as in [33, 35–38] noting that there is no codimension-2 contribution because the conformal factor is unity on the boundary, implying $\omega_k = 0$.

Then (20) can be written as

$$W_{ND}[\bar{g}, g] = \tfrac{1}{2}(W_D[\bar{g}, g] + \overline{W}_N[\bar{g}, g]), \tag{21}$$

where $\overline{W}_N$ means the usual Neumann expression, omitting the zero mode piece, and I can use the known values ($\bar{g}$ = disc and g = hemisphere),

$$W_D[\bar{g}, g] = \tfrac{1}{6}\log 2 - \tfrac{1}{3}, \qquad \overline{W}_N[\bar{g}, g] = \tfrac{2}{3}\log 2 + \tfrac{1}{6}. \tag{22}$$

The ζ-function on the ND-hemisphere follows from (15) with $\beta = \pi$. The ζ-function, $\zeta^{DD}_{\pi}(s)$ is the usual hemisphere ζ-function and the determinant has been considered a number of times. $\zeta^{DD}_{2\pi}(s)$, corresponds to Sommerfeld's double covering of three space introduced in connection with the half-plane boundary.

Since one needs conformal invariance in two dimensions, not three, the ζ-functions are actually *modified* Barnes ζ-functions which have been dealt with in [31, 39, 40]. The determinants can be computed generally in terms of Barnes ζ-functions but, because of the rational nature of π/β, in this case, they can be reduced to Epstein or Hurwitz ζ-functions. The general theory, appropriate to the arbitrary 2-lune, is developed in [31]. However, it is probably easier to proceed directly.

From [31] the ζ-function for $-\Delta$ on the ND 2-hemisphere is

$$\zeta_\pi^{ND}(s) = \zeta_{2\pi}^{DD}(s) - \zeta_\pi^{DD}(s) = \sum_{m,n=0}^{\infty} \frac{1}{((1+m+n)^2 - 1/4)^s}.$$

Expanding in the $1/4$ leads to the expression for the derivative at 0,

$$\zeta_\pi^{ND\prime}(0) = \zeta_2'(0, 1/2 \mid 1, 1) + \zeta_2'(0, 3/2 \mid 1, 1) - \frac{N_2(1)}{4}, \tag{23}$$

where

$$\zeta_2(s, a \mid 1, 1) = \sum_{m,n=0}^{\infty} \frac{1}{(a+m+n)^s}$$

is a two-dimensional Barnes ζ-function and $N_2(a)$ is its residue at $s = 2$; $N_2(a) = 1$.

In this simple case the sums can be easily rearranged,

$$\sum_{m,n=0}^{\infty} \frac{1}{(a+m+n)^s} = \sum_{N=0}^{\infty} \frac{N+1}{(N+a)^s} = \zeta_R(s-1, a) + (1-a)\zeta_R(s, a), \tag{24}$$

so that

$$\zeta_2(s, 1/2 \mid 1, 1) + \zeta_2(s, 3/2 \mid 1, 1) = 2\zeta_R(s-1, 1/2) = 2(2^{s-1} - 1)\zeta_R(s-1)$$

and therefore from (23)

$$\zeta_\pi^{ND\prime}(0) = -\zeta_R'(-1) - \tfrac{1}{12}\log 2 - \tfrac{1}{4}. \tag{25}$$

The absence of a $\zeta_R'(0)$ term is related to the absence of the perimeter heat-kernel coefficient caused by the equal-sized N and D regions. The ζ-function has only the Weyl volume pole.

For comparison the standard formulae for the DD and NN-hemispheres are

$$\zeta_\pi^{DD\prime}(0) = 2\zeta_R'(-1) - \zeta_R'(0) - \tfrac{1}{4}$$

and

$$\zeta_\pi^{NN\prime}(0) = 2\zeta_R'(-1) + \zeta_R'(0) - \tfrac{1}{4}.$$

By conformal transformation, on the ND-disc, the final result is

$$W_{ND}^{\text{disc}} = \tfrac{1}{2}\zeta_R'(-1) + \tfrac{11}{24}\log 2 - \tfrac{1}{24},$$

using (21) with (22).

7. The Robin boundary condition[1]

The Robin condition (11), has made only a formal appearance in the discussion so far. It was needed for conformal transformations but has not yet entered into any eigenproblem.

[1] According to Gustafson, [41, 42], Gustave Robin (1855–1897), never seemed to have used this condition. His name was, apparently, first attached to it by Bergmann and Schiffer in the 1950s but the condition had occurred already in the work of Newton [43].

The reason why Robin conditions are so awkward is that, in general, the problem does not separate and, even if it does, the eigenvalues are given only implicitly. Early considerations of the eigenproblem are reviewed by Pockels [44]. Poincaré [45] also used the condition in connection with eigenfunction existence. See also Bandle [46]. A practical, more recent, treatment is given by Strauss [47].

Apart from applied mathematics, there has been some recent interest in Robin conditions in the quantum field theoretic and spectral geometry scenes (e.g. Fulling [30], Bondurant and Fulling [48], Romeo and Saharian [49], Solodukhin [50], de Albuquerque and Cavalcanti [51]).

In this section I return to the 2-hemisphere, some aspects of which were mentioned in the previous section. Here I wish to see how far the standard Robin eigenproblem on the interval (e.g. [44, 47, 52]), is relevant for a spherical geometry.

Appropriate details of the classic hemisphere eigenproblem were also given in [53]. To repeat, coordinates on the hemisphere are $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant \pi$. The rim (boundary) consists of the union of the two semicircles $\phi = 0$ and $\phi = \pi$. In [53], D conditions were applied at $\phi = 0$, and N at $\phi = \pi$. Now the latter are replaced by Robin conditions and the former by either D or N. It is also possible to treat both Robin, but I will not make this generalization simply for convenience. The singular region, Σ, comprises just the S and N poles.

The Robin condition (11) specifically is

$$\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\bigg|_{\phi=\pi} = -S\Phi|_{\phi=\pi}, \tag{26}$$

which is not consistent with a separated structure for Φ unless S takes the form,

$$S = -\frac{h}{\sin\theta}, \tag{27}$$

diverging on Σ. I will return to this point later and take S as in (27) simply in order to get on with the calculation because, in this case, condition (26) reduces to the usual (D, R) (or (N, R)) on the ϕ 'interval' $(0, \pi)$ and I can employ known results. In the separated solution for Φ the θ part is unchanged, only the ϕ factor is modified. Thus, in the (D, R) case, the hemisphere eigensolution is

$$\Phi_\lambda = \mathcal{N}_h \sin(k\phi) P^{-k}_{n+k}(\cos\theta), \tag{28}$$

where $k > 0$ is determined by the one-dimension interval condition,

$$k\cot(k\pi) = h, \tag{29}$$

so that, as $h \to 0$, k tends to half an odd integer, $k \to m + 1/2$, $m = 0, 1, \ldots$, the Neumann result.

Likewise, for the (N, R) case, instead of (28) and (29) there is

$$\Phi_\lambda = \mathcal{N}_h \cos(k\phi) P^{-k}_{n+k}(\cos\theta) \tag{30}$$

and

$$k\tan(k\pi) = -h. \tag{31}$$

This time, as $h \to 0$, $k \to m$, $m = 0, 1, \ldots$. In both cases I will label k by the associated m.

To be specific, the eigenproblem I now wish to consider is

$$\left(-\Delta + \tfrac{1}{4}\right)\Phi_\lambda = \lambda\Phi_\lambda, \tag{32}$$

the reason being that the eigenvalues are perfect squares,

$$\lambda = \lambda_{mn} = \left(\tfrac{1}{2} + k_m + n\right)^2, \qquad m, n = 0, 1, \ldots. \tag{33}$$

Degeneracies, if there are any, are due to coincidences.

An important qualification must be made for (N, R) conditions with $h > 0$. In this case, k_0 is imaginary and the structure (33) does not apply with n integral. The precise analysis of this situation lies beyond the scope of this paper and so I will exclude the $m = 0$ contribution from the $h > 0$ expressions, which, it is stressed, are then valid only for the arbitrarily truncated sector for which m starts at 1 in (33). The discarded 'imaginary' interval mode remains to be incorporated.

One aim is to relate the spectral properties of the λ to those of the k_m by summing out the n. I will do this via the heat and cylinder kernels. We have used this before in spherical situations, [32]. The heat-kernel and cylinder (or 'square root') kernel are defined, in general, by

$$K(t) = \operatorname{Tr} e^{-Pt} = \sum_\lambda d_\lambda \, e^{-\lambda t}, \qquad K^{1/2}(t) = \operatorname{Tr} e^{-\sqrt{P}t} = \sum_\lambda d_\lambda \, e^{-\sqrt{\lambda}t}, \tag{34}$$

where I have included a degeneracy, just in case. From now on I use Fulling's notation, setting $T(t) \equiv K^{1/2}(t)$, and taking t as a generic parameter.

Using expression (33), it readily turns out that the hemisphere (HS) cylinder kernels factorize,

$$T_{\rm HS}(t) = \frac{1}{2\sinh t/2} T_I(t), \tag{35}$$

T_I being the cylinder kernel on the interval defined by,

$$T_I(t) = \sum_{m=m_0}^{\infty} e^{-k_m t}, \tag{36}$$

where $m_0 = 1$ for (N, R) with $h > 0$ and $m_0 = 0$ otherwise. Equation (35) is the connection between the hemisphere and the π-interval.

Important information is contained in the short-time expansions of the heat and cylinder kernels. A reflection of the pseudodifferential-operator character of $(-\Delta + 1/4)^{1/2}$ is the possible existence of logarithmic terms in the expansion of $T_{\rm HS}$.

It is relatively straightforward to show that on a d-dimensional manifold, maybe with a boundary, the expansion of a general cylinder kernel, T, takes the form,

$$T(t) \sim \sum_{i=0}^{\infty} a_i t^{-d+i} + \sum_{i}^{\infty} a'_i t^{-d+i} \log t. \tag{37}$$

The lower limit on the second term is deliberately left unspecified but I draw attention to the important fact that, if the operator P in (34) is a smooth differential operator with smooth boundary conditions, then only *odd* positive powers of t occur in the log term, whatever the dimension of the manifold.

One way of showing this is to use the known existence, in this case, of the series expansion of the heat-kernel $K(t)$, (1), and then, by ζ-function manipulations, relate the coefficients a, a' and C. This was first done in a physical context by Cognola *et al* [54]. A more recent analysis, in the compact case, has been performed by Bär and Moroianu [55] who consider local, diagonal kernels and give a careful analysis of estimates. Another way is to relate the asymptotic expansions obtained by smoothing using either λ or $\sqrt{\lambda}$ as the preferred variable. This was employed by Fulling [30]. It is sufficient to note, for now, that the coefficients of the logarithmic terms, a'_i in (37) are determined by the b_k.

The heat-kernel coefficients, b_k, for K_I, on the (R, R) interval have been obtained in [56] and show that there are logarithmic terms in T_I. Relation (35) between cylinder kernels then implies that $T_{\rm HS}$ also has logarithmic terms, but with *even* powers of t. In turn, this suggests that the operator $-\Delta + 1/4$, together with the boundary conditions, including the choice of Robin parameter, (27), is a rather singular operator. This is looked at further in the next section.

8. Asymptotic series

To allow for the fact that the operator $-\Delta + 1/4$ might be particularly singular, I generalize (1) to include logarithmic terms, the immediate aim being to relate the expansions of the heat and cylinder kernels. For this purpose, I use the zeta-function approach, mentioned earlier, together with the general series established by Grubb and Seeley [12]. Suitable summaries can be found in [57, 58]. For notational brevity I set $P = -\Delta + 1/4$ and $Q = \sqrt{P}$, but, so far as the general equations go, P can be any Laplace-like (second-order elliptic) operator of smooth form.

The asymptotic expansion of the heat-kernel $K(t)$, (34), is, e.g. [57, 58], on a d-dimensional manifold,

$$K(t) \sim \sum_{-d\leqslant k<-d+2} b_k t^{k/2} + \sum_{k=-d+2}^{\infty} (b'_k \log t + b''_k) t^{k/2}.$$

The reason for the lower limit of $-d + 2$ will appear later. This limit differs from that in [11, 58], which is zero.

The simple powers have been split into two sets because the coefficients have different qualities. Since this does not concern me at this time, I will combine them for algebraic ease. Therefore,

$$K(t) \sim \sum_{k=-d}^{\infty} b_k t^{k/2} + \sum_{k=-d+2}^{\infty} b'_k t^{k/2} \log t. \tag{38}$$

This generalizes (1), with the relation between the coefficients

$$b_k = \frac{C_{(k+d)/2}}{(4\pi)^{d/2}}. \tag{39}$$

Similarly for the cylinder kernel (Q is a first-order pseudo-operator) I will assume,

$$T(t) \sim \sum_{k=-d}^{\infty} a_k t^k + \sum_{k=-d+2}^{\infty} a'_k t^k \log t, \tag{40}$$

where now, *all* powers of t greater than $-d + 1$ appear in the logarithmic term.

The connection is made via the corresponding zeta-functions,

$$\zeta_Q(2s) = \zeta_P(s), \tag{41}$$

which have asymptotic expansions corresponding to (38) and (40). I refer to [57], e.g., in order to save work, and deduce,

$$\Gamma(s)\zeta_P(s) \sim \sum_{k=-d}^{\infty} \frac{b_k}{s + k/2} - \frac{n_0}{s} - \sum_{k=-d+2}^{\infty} \frac{b'_k}{(s + k/2)^2} \tag{42}$$

$$\Gamma(s)\zeta_Q(s) \sim \sum_{k=-d}^{\infty} \frac{a_k}{s + k} - \frac{n_0}{s} - \sum_{k=-d+2}^{\infty} \frac{a'_k}{(s + k)^2}, \tag{43}$$

where n_0 is the number of zero modes.

The relation between the a and the b follows from (41) using the standard duplication formula for $\Gamma(2s)$. Replacing s by $2s$ in (43) it is easy to see that dividing by $\Gamma(s+1/2)$ removes certain first-order poles and converts some second-order poles into first-order ones. From the residues, making the necessary identifications, I find the relations

$$\begin{aligned} b_k &= \frac{2^k\sqrt{\pi}}{\Gamma((1-k)/2)}a_k, \qquad k=-d,\ldots,-1,0,2,4,\ldots \\ &= (-1)^{(k+1)/2}2^{k-1}\Gamma((k+1)/2)\sqrt{\pi}a'_k, \qquad k=1,3,\ldots \\ b'_k &= \frac{2^{k-1}\sqrt{\pi}}{\Gamma((1-k)/2)}a'_k, \qquad k=-d+2,\ldots,-1,0,2,\ldots \\ &= 0 \qquad k=1,3,\ldots \end{aligned} \tag{44}$$

which generalize those found by Fulling [30] and Cognola *et al* [54].

Some overall conclusions can be drawn from these relations. Firstly, *given* the heat-kernel b_k, it is not possible to determine the cylinder a_k for odd positive, $k=1,3,\ldots$, as emphasized by Fulling. Secondly, for the assumed structure of the cylinder expansion (40), or (43), one cannot specify *all* the logarithmic terms in the heat kernel, i.e. all the b'_k. Since the form (40) is sufficient for the quantities appearing in this paper, I will leave this point except to say that it is easy to take (42) and work equation (41) the other way to derive the corresponding expansion for $\zeta_Q(s)$. The new feature is the appearance of poles of third order, leading to higher logarithmic powers in the cylinder kernel expansion.

The extension of the lower limit down to $-d+2$ has the rather violent consequence that, in this case, the coefficients b_i for $i\geqslant -d+2$ are global (i.e. nonlocal) quantities. Only b_{-d} and b_{-d+1} are locally computable. These are the Weyl volume and boundary area terms denoted by C_0 and $C_{1/2}$ earlier. This means, in particular that C_1, discussed in section 3, is not locally computable. We have seen that this is true for the exhibited form (13), through the last two, i.e. codimension 2, terms. The question of nonlocal terms is considered in a later section.

A further consequence of (44), and that is relevant for the spherical problem treated in the previous section, is that the logarithmic terms in the heat-kernel arise from those logarithmic terms in the cylinder kernel with *even* powers of t, which is precisely the case with (35).

The conclusion is that the Laplace operator, $-\Delta+1/4$ with the boundary conditions, *is* a rather singular operator as it provides a concrete example of a second-order problem involving logarithmic terms. It would therefore seem that it does not come within the compass of Seeley's analysis [17], probably because of the divergence in the Robin function, S. Hence, before giving explicit forms for the expansions, I break off to look at the modes, (28), a little more closely, one reason being that, although self-adjointness depends on the formal subtraction of two (identical) terms,

$$\int_0^\pi \mathrm{d}\theta\ \Phi S\Psi, \tag{45}$$

at the boundary, the divergence of S, (27), at the poles ($\theta=0,\pi$), i.e. on Σ, might give one pause for thought. In general terms, the assumed Dirichlet conditions at Σ are actually sufficient for convergence of the integral (45).

9. Robin mode properties

The orthonormality of (28), and of (30), is easily established either by direct integration or, formally, by the usual self-adjoint Liouville method.

For convenience I write down some standard things. For the Legendre functions orthonormality reads

$$\int_{-1}^{1} \mathrm{d}x\; P_{n+k}^{-k}(x) P_{n'+k}^{-k}(x) = \frac{2\Gamma(n+1)}{(2k+2n+1)\Gamma(2k+n+1)}\delta_{nn'},$$

where n and n' are positive integers or zero and $k > -1$ (MacRobert [59] p 335).

I also note the limiting behaviour at the poles,

$$P_{n+k}^{-k}(z) \to (1-z^2)^{k/2}\frac{1}{2^k\Gamma(k+1)}, \qquad z \to \pm 1. \tag{46}$$

The interval Robin modes are standard (e.g. Carslaw and Jaeger [52], Strauss [47], Pockels [44]) with easily determined normalizations. I again remark that Dirichlet conditions continue to apply on Σ, my default position.

Use of the limit (46) shows that each boundary term, (45), in the self-adjoint condition applied to two eigenfunctions, corresponding to k and k', is finite if $k + k' > 0$. So we are completely safe in this case.

This limit also implies that the total heat flux, per mode,

$$\int_{\partial\mathcal{M}} \partial_n \Phi_\lambda,$$

is finite. Moreover, the 'quantum mechanical flux',

$$\int_{\partial\mathcal{M}} \Phi_\lambda \partial_n \Phi_\lambda,$$

is also finite, by virtue of Barnes' formula,

$$\int_0^1 \mathrm{d}x \frac{\left(P_\nu^\mu(x)\right)^2}{1-x^2} = -\frac{1}{2\mu}\frac{\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)}, \tag{47}$$

valid for $\operatorname{Re}\mu < 0$ and $\mu+\nu$ a positive integer, or zero. The lower limit can be extended to -1 using the fact that the Legendre functions are unchanged, up to a sign, under $x \to -x$ ([59], p 334 examples (2) and (3)).

Confidence in the eigenmodes is increased if one employs a perturbation technique to calculate the change in the eigenvalues λ_{mn} of (33) for a small change in S. The formula, which will not be derived here, is

$$\delta\lambda = \int_{\partial\mathcal{M}(N)} \Phi_\lambda S \Phi_\lambda. \tag{48}$$

For simplicity, I am considering the (D, R) set-up and am perturbing about the (D, N) case, so S is small. The integration over the boundary encompasses only the part on which R (equivalently (N)) holds, since Φ is zero on the D part. Equation (48) seems to occur first, for constant S, in Poincaré [45] and for variable S in Pockels [44] p 178, as a quick consequence of Green's formula. See also Fröhlich [60], equation (6d).

The answer is known from a direct analysis of the interval eigenvalue condition (29) which shows that

$$k_m - \frac{1}{2} \sim m - \frac{2h}{(2m+1)\pi}. \tag{49}$$

Alternatively, applying (48) yields

$$\delta\lambda_{mn} = -h\mathcal{N}^2 \int_0^\pi \mathrm{d}\theta \frac{1}{\sin\theta}\left(P_{n+\overline{m}/2}^{-\overline{m}/2}(\cos\theta)\right)^2, \tag{50}$$

where $\overline{m} = 2m + 1$. The normalization is

$$\mathcal{N}^2 = \frac{2}{\pi}\frac{\Gamma(\overline{m}+n+1)\Gamma(\overline{m}+2n+1)}{2\Gamma(n+1)},$$

and substitution of (47) into (50) easily produces

$$\delta\sqrt{\lambda}_{mn} = -\frac{2h}{\pi}\frac{1}{\overline{m}},$$

which is just (49). This limited check inspires a certain confidence in the sensibility of the modes and the model.

I should point out now that I do not attach any serious significance to the model. The choice S is simply one of convenience for solvability.

10. Explicit expressions

In this section I use the form of the heat-kernel coefficients for the interval (R, R) Robin problem derived in [56], from which one can easily deduce those for the (D, R) and (N, R) cases by appropriate limits.

I find, for both (N, R) $(h < 0)$ and (D, R) on an interval of length π, (I), the heat-kernel coefficients

$$b_k = b_k^I = \frac{1}{2\Gamma(k/2+1)}h^k, \qquad k = 1, \ldots, \tag{51}$$

so that, for example, the coefficients of the logarithmic terms in the interval *cylinder* kernel, (36), are determined from (44) applied to the interval, as

$$a'_{2n-1} = (-1)^n\frac{1}{\pi}\frac{1}{(2n-1)!}h^{2n-1}, \qquad n = 1, 2, \ldots,$$

and these odd terms are all there because there are no logarithmic terms in the interval heat-kernel. The asymptotic series can be summed to produce the closed form for the asymptotic logarithmic part of T_I,

$$T_I^{\log}(t) \sim \frac{\mathrm{e}^{-h^2t^2}-1}{\pi ht}\log t,$$

and then from (35) the logarithmic part of the (N, R) $(h < 0)$, or (D, R), hemisphere, (HS), cylinder kernel follows as

$$T_{\mathrm{HS}}^{\log}(t) \sim \frac{1}{\sinh(t/2)}\frac{\mathrm{e}^{-h^2t^2}-1}{2\pi ht}\log t. \tag{52}$$

The same result also holds for the truncated $(N, R), h > 0$, sector, mentioned before. The removal of the contribution of the imaginary k_0 mode to the heat-kernel coefficients does not affect the b_k^I, for odd k.

Relation (44) can now be applied to the hemisphere and the logarithmic terms in the *heat-kernel* worked out. I do not give the easily derived expressions. They are combinations of Bernoulli numbers. Their existence is all that is required for now. I only remark that the important term $\sim t^0 \log t$ is present.

There are, of course, a number of technical routes to the expressions and conclusions derived above. I have chosen to use heat and cylinder kernels but one could employ the resolvent and a standard contour method of rewriting eigenvalue sums. I also note that it is straightforward to extend the calculations to a lune and also to d-dimensions, which just gives higher powers of $\sinh(t/2)$ in (52), say. This justifies the lower limit in (40) and hence in (38). Odd spheres are not singular, in agreement with a general result. The details will be presented at another time.

11. Nonlocal terms and the Casimir energy

In this section I enlarge on previous statements regarding the nonlocality of some expansion coefficients, in particular of C_1 which, in two dimensions, is the coefficient of the constant term in the heat-kernel expansion.

As a first step, I look at the Casimir energy, E, on the π-interval (I) and simply quote the formal definition,

$$E = \mathrm{FP}\tfrac{1}{2}\zeta_P^I(-1/2), \tag{53}$$

where $P = -\mathrm{d}^2/\mathrm{d}\phi^2$ and the boundary conditions are either (D, R) or (N, R) on the ends. FP stands for 'finite part'. The Casimir energy is a nonlocal quantity.

On the π-interval the relevant definition here of the ζ-function is

$$\zeta_P^I(s) = \sum_{m=m_0} \frac{1}{k_m^{2s}}. \tag{54}$$

For the purposes of this paper, I consider E, defined by (53), simply as a convenient mathematical quantity rather than as something having physical, operational significance.

Equations (41), (42) and (43) hold for the interval. Since P is smooth, there are no log terms in the heat kernel and so the coefficients $b_k'^I$ are zero.

It is important to realize that in the Robin case, $\zeta_P^I(s)$ has a pole at $s = -1/2$,

$$\zeta_P^I(s) \sim \frac{A}{s+1/2} + B \tag{55}$$

where the residue, $A = -b_1^I/2\sqrt{\pi}$, follows from (42) and the remainder, B, equals $2E$, by definition. In terms of the interval heat-kernel coefficients, C_n^I, (see (51), (39)),

$$b_1^I = \frac{1}{2\sqrt{\pi}} C_1^I = \frac{h}{\sqrt{\pi}}, \tag{56}$$

for both (D, R) and (N, R), for all h.

From (53) and (41) it is required to work around $s \sim -1$ for $\zeta_Q^I(s)$ where

$$\Gamma(s)\zeta_Q^I(s) \sim \frac{a_1^I}{s+1} - \frac{a_1'^I}{(s+1)^2} \tag{57}$$

and so

$$\left(-\frac{1}{s+1} + \gamma - 1\right)\left(\frac{A}{s/2+1/2} + B\right) \sim \frac{a_1^I}{s+1} - \frac{a_1'^I}{(s+1)^2},$$

which yields

$$b_1^I = -\sqrt{\pi}\, a_1'^I, \tag{58}$$

and also

$$E = -\frac{1}{2}a_1^I + \frac{\gamma-1}{2\sqrt{\pi}} b_1^I = -\frac{1}{2}a_1^I - \frac{\gamma-1}{2\pi} h. \tag{59}$$

Equation (58) agrees with (44) while (59) relates the coefficient a_1^I to the nonlocal Casimir energy. It differs from the result in [30] by the last (constant) term. See [54] for a relevant discussion of various regularization recipes in this context.

The next step is to relate the interval and hemisphere expansions using (35). Expansion of the $1/\sin h$ easily gives the connection,

$$a_0^{\mathrm{HS}} = a_1^I - \tfrac{1}{24} a_{-1}^I, \tag{60}$$

which I now rewrite in terms of heat-kernel expansion coefficients.

Application of (44) to the two-hemisphere gives,

$$a_0^{\mathrm{HS}} = b_0^{\mathrm{HS}},$$

where b_0^{HS} is the constant term coefficient,

$$b_0^{\mathrm{HS}} = \frac{1}{4\pi} C_1^{\mathrm{HS}}.$$

Equation (44), applied to the interval, gives

$$a_{-1}^{I} = \frac{2}{\sqrt{\pi}} b_{-1}^{I} = \frac{2}{\sqrt{\pi}} \frac{C_0^I}{2\sqrt{\pi}} = 1,$$

and so, finally, (60) becomes, using (59),

$$C_1^{\mathrm{HS}} = -8\pi E - \frac{\pi}{6} - 4h(\gamma - 1), \tag{61}$$

which relates the C_1 coefficient on the hemisphere to the Casimir energy on the interval *for my model*, i.e. (27).

A basic check sets $h = 0$ when (61) allows a computation of the interval Casimir energies for the (D, D) and (D, N) cases using the expressions for C_1, (7), (12) and (13). Simple algebra yields the standard values,

$$E(D, D) = E(N, N) = -\tfrac{1}{24}, \qquad E(D, N) = \tfrac{1}{48}.$$

It is worth noting that for the (D, D) and (N, N) cases, although the Casimir energy, E, is nonlocal on the interval, the C_1 coefficients are local on the hemisphere.

In the general case, E is a nontrivial function of h on the interval and the conjectured forms of $C_1(D, R)$ and $C_1(N, R)$, (13), (8), obviously do not agree with (61). However, it will be recalled that the heat-kernel expansion is really one in $h^2 t$ and therefore one should treat h as 'small'. As mentioned, it is in this realm that (13) and (12) should be valid. Furthermore the divergence of the hemisphere Robin function, S of (27), makes C_1 formally infinite, and cannot be considered 'small'. Hence it is clear that (13), (12) and (8) are not even appropriate for the present situation. Against this must be set the fact that, as described in section 9, perturbation in S appears to work for the eigenvalues. Therefore I intend to give a further look at perturbation theory. This will give us a simple, if limited, analytical handle on the ζ-function on the Robin interval which might also be useful in other circumstances.

12. Perturbation approach

I first consider the (N, R) case when, as discussed earlier, one must distinguish between positive and negative Robin parameters, h. If $h > 0$ one has the approximation for the considered interval frequencies,

$$k_m \approx m - \frac{h}{m\pi}, \qquad m = 1, 2, \ldots,$$

which leads to the ζ-function for the truncated theory,

$$\zeta_P^I(s) \approx \zeta_R(2s) + \frac{2hs}{\pi} \zeta_R(2s + 2), \tag{62}$$

in terms of the Riemann ζ-function, and so $A = -h/2\pi$ with

$$E(N, R) \approx -\frac{1}{24} - \frac{h}{2\pi}(\gamma - 1), \qquad h \downarrow 0. \tag{63}$$

If $h < 0$, there is a real root which tends to zero as $h \to 0$ and becomes the zero (N, N) mode. Approximation of (31) gives,

$$k_0 \approx \sqrt{\frac{-h}{\pi}}, \qquad h \uparrow 0, \qquad k_m \approx m - \frac{h}{m\pi}, \qquad m = 1, 2, \ldots,$$

whence $A = -h/2\pi$ and

$$E(N, R) \approx -\frac{1}{24} + \frac{1}{2}\sqrt{\frac{-h}{\pi}} - \frac{h}{2\pi}(\gamma - 1), \qquad h \uparrow 0. \tag{64}$$

Turning to the (D, R) case, to order h, (49),

$$k_m \approx \frac{\overline{m}}{2} - \frac{2h}{\overline{m}\pi} \qquad \text{i.e.} \qquad \lambda \approx \frac{\overline{m}^2}{4} - \frac{2h}{\pi}, \qquad \overline{m} = 1, 3, \ldots,$$

so that

$$\zeta_P^I(s) \approx (2^{2s} - 1)\zeta_R(2s) + \frac{2hs}{\pi}(2^{2s+2} - 1)\zeta_R(2s + 2). \tag{65}$$

Working around $s \sim -1/2$ yields, in accordance with (55), the residue check, $A = -h/2\pi$, and the Casimir energy,

$$E(D, R) \equiv \frac{B}{2} \approx \frac{1}{48} - \frac{h}{2\pi}(\gamma - 1 + 2\log 2). \tag{66}$$

From these expressions, I use (61) to *compute* the heat-kernel coefficients on the hemisphere. I find, in the two cases, the values,

$$\begin{aligned}
C_1(N, R) &\approx \frac{\pi}{6}, \qquad h > 0\\
C_1(N, R) &\approx \frac{\pi}{6} - 4\sqrt{-\pi h}, \qquad h < 0\\
C_1(D, R) &\approx -\frac{\pi}{3} + 8h\log 2.
\end{aligned} \tag{67}$$

The asymmetry between positive and negative h for (N, R) can be traced to the omission of the imaginary mode for $h > 0$. It is conjectured that suitable reinstatement of this mode will restore the symmetry.

The log 2 term in $C_1(D, R)$ reinforces the conclusion that (13) is inappropriate for the present singular model.

Similar results can be shown to hold for the simple wedge of section 2 with Robin conditions on one side. In order for the techniques of [24] to work, separability demands that the Robin function is again singular on Σ (the apex of the wedge) taking the form $S = -h/r$. Without going into details, just referring to equations (4.4)–(4.6) of [24], there is again a log term coming from the pole at $s = -1/2$ in the interval Robin ζ-function and there is a log 2 term in the expression for the corresponding C_1.

13. Exact form of interval Casimir energy

The exact expressions for the Casimir energy derived in [49], [51] can be approximated for comparison with my perturbation results. For convenience I refer to equations (34) and (39) in [51] for the Casimir energies and rewrite them in one dimension,

$$\begin{aligned}
\mathcal{E}(N, R) &= E(N, N) + \frac{1}{2\pi}\int \mathrm{d}k \log\left(1 - \frac{2h}{(k - h)(\exp(2k\pi - 1))}\right)\\
\mathcal{E}(D, R) &= E(D, N) + \frac{1}{2\pi}\int \mathrm{d}k \log\left(1 + \frac{2k}{(k - h)(\exp(2k\pi - 1))}\right) - \frac{1}{16},
\end{aligned} \tag{68}$$

where the $\mathcal{E}$ may, or may not, agree, up to renormalization, with the quantities evaluated from (53). (I have set $h = -c_2 < 0$, in the notation of [51], to comply with my sign.)

The leading small h behaviours of the 'correction' terms in (68) can be determined *numerically* to be

$$\begin{aligned}\mathcal{E}(N, R) &\approx E(N, N) + \frac{1}{2}\sqrt{\frac{-h}{\pi}} + \frac{h}{2\pi}F_N(h)\\ \mathcal{E}(D, R) &\approx E(D, N) + \frac{h}{2\pi}F_D(h),\end{aligned} \tag{69}$$

where F_N and F_D satisfy the functional relation

$$F(\lambda x) - \lambda F(x) \approx \frac{\log\lambda}{\lambda}. \tag{70}$$

There is some similarity between (69) and the perturbative results, but I cannot explain (70) from such a viewpoint and simply present the result as a possible significant curiosity.

14. Comments and conclusion

Apart from rectilinear domains, and the hemisphere, there seem few situations that can be solved exactly for ND-conditions (see [44]) and this is a drawback to the construction of the precise forms of the heat-kernel coefficients. Nevertheless a certain progress has been made in a simple minded way making use of the ND-wedge expression. This type of reasoning can be extended to higher dimensions leading to information about the trihedral corner contributions and their higher analogues.

Surprisingly the conformal functional determinant is available on the 'half-N half-D' disc by conformal transformation from the ND-hemisphere, and has been computed, assuming that a conjecture for the heat-kernel coefficient, C_1, is correct.

In sections 7 to 10, I considered hybrid (D, R) and (N, R) 2-hemisphere problems, with a Laplace style operator, which have logarithmic terms in the short-time expansions of the heat kernels. The boundary conditions involve a Robin function that diverges, but not too strongly, at the poles which are the places where the D, N conditions meet the R condition. This allows the problem to be separated.

Although the second-order differential operator (with the boundary conditions) is rather singular, it is not singular enough to prevent the heat-kernel expansion from existing.

The (singular) Robin condition is essential here for the existence of the logarithmic terms. Expression (52) vanishes when $h = 0$. In [53] it was suggested, by an indirect argument, that for an (N, D) situation, a nonzero extrinsic curvature at the boundary gave rise to logarithmic terms. In general, it seems that the Robin condition mimics an extrinsic curvature [30] equation (4.1). If this is so, it is perhaps not surprising that a singular Robin function, S, produces a log term as it would simulate a conical-type singularity.

The attempt to relate the hemisphere heat-kernel coefficient to the Robin Casimir energy on the interval, analysed in sections 11 and 12, confirms the limited validity of the expression for the C_1 coefficient, (13), to extend which requires further analysis.

On a technical level it is noted that the (N, R), $h > 0$, results need to be completed by the inclusion of the effect of the imaginary modes, $m = 0$ in (33).

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